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ESTIMATES FOR THE JUNG CONSTANT IN BANACH LATTICES

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ABSTRACT

Sharp lower estimates for the Jung constant $J(E)$ in Banach lattices E satisfying an upper p-estimate and a lower q-estimate are given. Moreover, the minimal value of $J(E)$ with respect to equivalent renormings of E is calculated in $E = L_{p,q}$ for finite p and q, as well as in more general spaces E. Finally, a nontrivial estimate for the radius $r_{L_{p,\infty}}(A)$ is obtained for A being a bounded sequence of disjointly supported functions in $L_{p,\infty}$.

Given a bounded set A in a Banach space E, let

$$d_E(A) = \sup_{x,y\in A} \|x-y\|, \quad r_E(A) = \inf_{y\in E} \sup_{x\in A} \|x-y\|$$

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denote the diameter and the radius, respectively, of A in E. The number

$$J(E) = \sup\{2r_E(A) : A \subset E,\, d_E(A) \le 1\}$$

is called the **Jung constant** of E. (In the Russian literature, e.g. [19, 23], the number $\frac{1}{2}J(E)$ is called Jung constant.) The study of this and related constants was initiated by Jung [12] almost 100 years ago and is still of interest in various fields of Banach space geometry and fixed point theory.

It is clear that $1 \le J(E) \le 2$ for any space E. Moreover, $J(E) = 1$ if and only if E is a so-called **P_1-space** [6]; in particular, $J(L_\infty) = J(l_\infty) = 1$.

By classical results [4],

$$J(l_2^n) = \sqrt{\frac{2n}{n+1}}.$$

The constant $J(l_1^n)$ was found for some n in [7]. A lower estimate for $J(L_p)$ was obtained in [1]. Moreover, it was proved in [2, 19] that

$$J(L_p) = J(l_p) = \max\{2^{1/p}, 2^{1/p'}\} \quad (1 \le p < \infty).$$

(Here and throughout in the sequel we write $p' = p/(p-1)$ for $1 < p < \infty$ and $1' = \infty$.) Finally, some estimates for the Jung constant for spaces with a symmetric basis were found in [10], and for Orlicz spaces in [22]. The survey [18] is dedicated to some geometric characteristics related to the Jung constant, while the recent survey [8] discusses interesting connections with fixed point theory for nonexpansive maps.

As a matter of fact, computing the exact value of the Jung constant in a specific Banach space is a highly nontrivial problem. Some results about Jung constants in rearrangement invariant spaces (see below) were announced in [11] and proved in [23]. In this paper we give sharp lower estimates for the Jung constant in the class of Banach lattices satisfying an upper p-estimate and a lower q-estimate for some $p, q \in [1, \infty)$ (Theorem 1 and Corollary 1). Moreover, we compute the minimal value of $J(L_{p,q})$ with respect to equivalent renormings of $L_{p,q}$ in case $q < \infty$ (Theorem 2), as well as in some more general spaces (Theorem 3). Explicit formulas or estimates for this minimal value are given for general reflexive spaces (Corollary 2), for so-called r-convexifications (Corollary 3), for reflexive Orlicz spaces (Corollary 4), and for nontrivial intersections of Lebesgue spaces (Corollary 5). The calculation of the Jung constant $J(L_{p,\infty})$ is an open problem. However, we obtain a nontrivial estimate for $r_{L_{p,\infty}}(A)$ if A is a bounded sequence of disjointly supported functions in $L_{p,\infty}$ (Theorem 4). This statement was announced without proof in [11].

Let us present the necessary definitions. Given $1 \le p \le \infty$, we say that a Banach lattice E satisfies an **upper p-estimate** if there exists a constant $C > 0$ such that, for every integer n and every choice of disjointly supported elements $x_1, x_2, \dots, x_n$ in E, we have

$$\left\| \sum_{i=1}^{n} x_i \right\| \le C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}, \tag{1}$$

with obvious modifications for $p = \infty$. Likewise, a **lower q-estimate** ($1 \le q \le \infty$) has the form

$$\left\| \sum_{i=1}^{n} x_i \right\| \ge c \left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \tag{2}$$

for some constant $c > 0$. A Banach lattice E is called **p-convex** if there exists a constant $C > 0$ such that, for every integer n and every choice of elements $x_1, x_2, \dots, x_n$ in E, we have

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \le C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}. \tag{3}$$

Similarly, we call E **q-concave** ($1 < q < \infty$) if

$$\left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} \right\| \ge c \left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q}. \tag{4}$$

for some constant $c > 0$.

Given a pair of Banach function spaces (E, F) over the same domain, the **Calderón construction** $[E, F]_\theta$ ($0 < \theta < 1$) is, by definition, the set of all measurable functions for which the expression

$$\|x\|_{[E,F]_\theta} = \inf_{\|u\|_E, \|v\|_F \le 1} \sup_t \frac{|x(t)|}{|u(t)|^{1-\theta} |v(t)|^\theta}$$

is finite.

Recall that a Banach space E of measurable functions on $[0, 1]$ with the Lebesgue measure is called **rearrangement invariant** (r.i.) or **symmetric** if $x^* \le y^*$, $y \in E$ implies $x \in E$ and $\|x\| \le \|y\|$; here x^* denotes the decreasing rearrangement of $|x|$. Following [15] we shall assume throughout that E is separable or isometric to a conjugate space.

If two r.i. spaces coincide as sets, then their norms are equivalent. It is evident that the Jung constant is not stable with respect to equivalent renormings. Therefore it seems interesting to consider the characteristic

$$J_0(E) = \inf J(E),$$

where the infimum is taken over all equivalent r.i. norms on E. In [23] it was proved that $J_0(L_p) = J(L_p)$ for every $p \in (1, \infty)$.

The following two examples which both generalize the Lebesgue space L_p will be useful in what follows. Given $1 < p < \infty$ and $1 \leq q \leq \infty$, the **Lorentz space** $L_{p,q}$ consists of all measurable functions for which the expression

$$\|x\|_{L_{p,q}} = \begin{cases} \left(\frac{q}{p} \int\limits_0^1 [x^*(t) t^{1/p}]^q \frac{dt}{t} \right)^{1/q} & \text{for} \quad q < \infty \\ \sup\limits_{0<t\leq 1} x^*(t) t^{1/p} & \text{for} \quad q = \infty \end{cases}$$

is finite. This expression is a norm if $q \leq p$, and a quasinorm if $q > p$. In the second case, $\|x\|_{p,q}$ is equivalent to a norm. Of course, $L_{p,p}$ is isomorphic to L_p, since the map $x \mapsto x^*$ is an L_p-isometry.

The other example is the **Orlicz space** L_M which consists of all measurable functions x for which the (Luxemburg) norm

$$\|x\|_{L_M} = \inf \{k : k > 0, \int_0^1 M[x(s)/k] \, ds \leq 1\}$$

is finite. Here $M\colon \mathbb{R} \to \mathbb{R}$ is a given Young function [13, 21]; the special choice $M(u) = |u|^p$ gives, of course, again the Lebesgue space L_p.

All definitions and results about Banach lattices and r.i. spaces mentioned above may be found, together with numerous examples, in the monographs [3, 14, 15].

Before stating our first theorem we have to prove some auxiliary results.

LEMMA 1: *Let a_{ij} $(i,j = 1,2,\ldots)$ be a symmetric bounded non-negative sequence with $a_{ii} = 0$, and let $\varepsilon > 0$. Then there exists a strictly increasing sequence $\{k_i\}$ in $\mathbb{N}$ such that*

$$\sup_{i,j \in \mathbb{N}} a_{k_i,k_j} \leq \inf_{i \in \mathbb{N}} a_{k_{2i-1},k_{2i}} + \varepsilon.$$

Proof: Put

$$\gamma = \inf_{\substack{I \subseteq \mathbb{N} \\ |I| = \infty}} \sup_{i,j \in I} a_{ij}.$$

We can find a set $I_1 \subseteq \mathbb{N}$, $|I_1| = \infty$, for which

$$\gamma \le \sup_{i,j\in I_1} a_{ij} < \gamma + \frac{\varepsilon}{2}.$$

Then, for every $J \subseteq I_1$, $|J| = \infty$, we have

$$\gamma \le \sup_{i,j\in J} a_{ij} < \gamma + \frac{\varepsilon}{2}. \tag{5}$$

Choose a pair $k_1, k_2 \in I_1$ such that $k_1 < k_2$ and

$$\gamma - \frac{\varepsilon}{2} < a_{k_1,k_2} < \gamma + \frac{\varepsilon}{2}.$$

Applying (5) to the set $I_2 = I_1 \backslash \{k_1, k_2\}$, we find $k_4 > k_3 > k_2$ such that

$$\gamma - \frac{\varepsilon}{2} < a_{k_3,k_4} < \gamma + \frac{\varepsilon}{2}.$$

Continuing this way, we obtain a strictly increasing sequence $\{k_i\}$ such that

$$\gamma - \frac{\varepsilon}{2} < a_{k_{2i-1},k_{2i}} < \gamma + \frac{\varepsilon}{2}$$

for every $i \in \mathbb{N}$. Consequently,

$$\sup_{i,j\in\mathbb{N}} a_{k_i,k_j} \le \sup_{i,j\in I_1} a_{i,j} < \gamma + \frac{\varepsilon}{2} < \inf_{i\in\mathbb{N}} a_{k_{2i-1},k_{2i}} + \varepsilon$$

as claimed. ∎

Consider the Walsh matrices

$$W_0 = (1), \quad W_n = \begin{pmatrix} W_{n-1} & W_{n-1} \\ W_{n-1} & -W_{n-1} \end{pmatrix} \qquad (n = 1, 2, \ldots).$$

LEMMA 2: *For $n \in \mathbb{N}$, let $\{y_k\}$ $(k = 1, 2, \ldots, 2^n)$ be a sequence of disjointly supported elements of a Banach lattice E, and put*

$$z_i = \sum_{k=1}^{2^n} w_{ik} y_k \qquad (i = 1, 2, \ldots, 2^n),$$

where $\{w_{ik}\} = W_n$. If E does not contain c_0, then the set $B_n = \{z_1, \ldots, z_{2^n}\}$ satisfies

$$r_E(B_n) = \|y_2 + y_3 + \cdots + y_{2^n}\|.$$

Proof: For $i = 1, 2, \ldots, 2^n$ we have $\|z_i - y_1\| = \|y_2 + y_3 + \cdots + y_{2^n}\|$, hence

$$r_E(B_n) \le \|y_2 + y_3 + \cdots + y_{2^n}\|.$$

Suppose that

$$r_E(B_n) < ||y_2 + y_3 + \cdots + y_{2^n}||.$$

Then there exists $z \in E$ such that

$$||z_i - z|| < ||y_2 + y_3 + \cdots + y_{2^n}||$$

for $i = 1, 2, \ldots, 2^n$. Let P_i denote the band projection (see Section 1.b of [15]) corresponding to y_i $(i = 1, 2, \ldots, 2^n)$. Since $P_1 z_i = y_1$ for $i = 1, 2, \ldots, 2^n$, we may suppose that $P_1 z = y_1$. It is clear that $||w_{i1}P_1 + \cdots + w_{i2^n}P_{2^n}|| = 1$, hence

$$||z_i - z|| = \left\| \sum_{k=1}^{2^n} w_{ik} y_k - z \right\| \geq \left\| \left(\sum_{k=1}^{2^n} w_{ik} P_k \right) \left(\sum_{k=1}^{2^n} w_{ik} y_k - z \right) \right\|.$$

$$= \left\| \sum_{k=2}^{2^n} y_k - \sum_{k=2}^{2^n} w_{ik} P_k z \right\|$$

for $i = 1, 2, \ldots, 2^n$. From

$$\sum_{i=1}^{2^n} \sum_{k=2}^{2^n} w_{ik} P_k z = \sum_{k=2}^{2^n} P_k z \sum_{i=1}^{2^n} w_{ik} = 0$$

we conclude that

$$2^n \left\| \sum_{k=2}^{2^n} y_k \right\| = \left\| \sum_{i=1}^{2^n} \left(\sum_{k=2}^{2^n} y_k - \sum_{k=2}^{2^n} w_{ik} P_k z \right) \right\|$$

$$\leq \sum_{i=1}^{2^n} \left\| \sum_{k=2}^{2^n} y_k - \sum_{k=2}^{2^n} w_{ik} P_k z \right\| = \sum_{i=1}^{2^n} ||z_i - z|| < 2^n \left\| \sum_{k=2}^{2^n} y_k \right\|.$$

This contradiction proves our assertion. ∎

We remark that a somewhat weaker form of Lemma 2 was proved in [23].

Let E be a Banach lattice. Following B. Maurey [17], for $n \in \mathbb{N}$ we set

$$V_n(E) = \inf \left\{ \left\| \sum_{i=1}^{n} x_i \right\| : ||x_i|| = 1; \;\; x_i \wedge x_j = 0 \text{ for } i \neq j \right\}.$$

Maurey has proved that either $V_n(E) = 1$ for each n or

$$\lim_{n \to \infty} V_n(E) = \infty. \tag{6}$$

It is known (see Propositions 1.f.7 and 1.f.12 of [15]) that (6) is equivalent to the q-concavity of E for some $q < \infty$; this is in turn equivalent to the fact that E satisfies a lower q-estimate for some $q < \infty$. We also set

$$\overline{\lim_{n \to \infty}} \sqrt[n]{V_{2^n}(E)} = Q(E).$$

Note that always $1 \le Q(E) \le 2$.

THEOREM 1: *Suppose that E is a Banach lattice which does not contain a subspace isomorphic to c_0. Then for any infinite sequence $B = \{b_i\}$ of disjointly supported elements of E we have*

$$r_E(B) \ge \overline{\lim_{n\to\infty}} \|b_n\|. \tag{7}$$

If $\{y_i\}$ is a sequence of normalized disjointly supported elements of E one has

$$J(E) \ge \overline{\lim_{n\to\infty}} \Big\|\sum_{i=1}^{2^n} y_i\Big\|^{1/n} \ge Q(E). \tag{8}$$

Moreover, if $\{x_i\}$ is a sequence of normalized disjointly supported elements of E, then for each n there exists an $I_n \subset \mathbb{N}$ with $|I_n| = 2^n$ such that

$$J(E) \ge 2 \overline{\lim_{n\to\infty}} \Big\|\sum_{j\in I_n} x_j\Big\|^{-1/n}. \tag{9}$$

Finally, we have in fact $J(E) \ge \sqrt{2}$.

Proof: Assume that $r_E(B) = r < k = \overline{\lim}\|b_n\|$. For $0 < \varepsilon < (k-r)/2$ there exists c_ε such that $\|b_n - c_\varepsilon\| < k - 2\varepsilon$ for all $n \in \mathbb{N}$. Since E does not contain c_0, there exists a band projection of c_ε on b_n which we denote by z_n. We have then

$$\|b_n\| \le \|b_n - z_n\| + \|z_n\| \le \|b_n - c_\varepsilon\| + \|z_n\| < k - 2\varepsilon + \|z_n\|,$$

i.e. $\|z_n\| > \|b_n\| - k + 2\varepsilon$. Choose a subsequence $\{n_j\}$ in $\mathbb{N}$ such that $\|b_{n_j}\| > k - \varepsilon$, hence $\|z_{n_j}\| > \varepsilon$ for $j \in \mathbb{N}$. Then we have

$$\Big\|\sum_{j=1}^{m} \frac{z_{n_j}}{\|z_{n_j}\|}\Big\| \le \frac{1}{\varepsilon}\|c_\varepsilon\|$$

for every $m \in \mathbb{N}$, which means that the sequence $\|z_{n_j}\|^{-1} z_{n_j}$ is equivalent to the standard basis of c_0. This contradiction proves (7).

Let us now prove (8). First of all, we claim that $J(E) > 1$. In fact, if $J(E) = 1$, then E is a P_1 space which implies that it is isometric to $C(K)$ with K being some extremely disconnected set. Because E is infinite dimensional, this implies that E contains c_0, contradicting our hypothesis. Thus, we have proved that $J(E) > 1$.

To prove (8), we use the notation of Lemma 2. Since

$$\#\{k : 1 \le k \le 2^n,\, w_{ik} = w_{jk}\} = \#\{k : 1 \le k \le 2^n,\, w_{ik} = -w_{jk}\} = 2^{n-1}$$

for every choice of $i, j \in \{1, 2, \ldots, 2^n\}$ with $i \neq j$, we have

$$d_E(B_n) = 2\max\left\{\Big\|\sum_{k\in I} y_k\Big\| : |I| = 2^{n-1}\right\},$$

where the maximum runs over all index sets $I \subset \{1, 2, \ldots, 2^n\}$ with precisely 2^{n-1} elements. By Lemma 2, we may choose such an index set I_n with

$$\Big\|\sum_{i=1}^{2^n} y_i\Big\| \leq 1 + r_E(B_n) \leq 1 + \frac{1}{2}J(E)d_E(B_n) \leq 1 + J(E)\Big\|\sum_{k\in I_n} y_k\Big\|.$$

Iterating this inequality we get

$$\Big\|\sum_{i=1}^{2^n} y_i\Big\| \leq 2\sum_{k=0}^{n-1} J^k(E) = 2\frac{J^n(E)-1}{J(E)-1} \leq \frac{2J^n(E)}{J(E)-1} \tag{10}$$

which implies (8).

Finally, to prove (9) fix $\varepsilon > 0$ and set $A_0 = \{x_n\}$. Applying Lemma 1 to the sequence $a_{ij} = ||x_i - x_j||$ we see that there exists a subsequence $A_1 = \{u_n^{(1)}\}$ of A_0, where $u_n^{(1)} = x_n^{(1)}$, with the property that

$$d_E(A_1) \leq \inf_n ||x_{2n-1}^{(1)} - x_{2n}^{(1)}|| + \varepsilon = \inf_n ||v_n^{(1)}|| + \varepsilon,$$

where $v_n^{(1)} = x_{2n-1}^{(1)} - x_{2n}^{(1)}$. Putting $B_1 = \{v_n^{(1)}\}$, we find a subsequence $A_2 = \{u_n^{(2)}\}$ of B_1 with the property that

$$d_E(A_2) \leq \inf_n ||u_{2n-1}^{(2)} - u_{2n}^{(2)}|| + \varepsilon = \inf_n ||v_n^{(2)}|| + \varepsilon$$

where $v_n^{(2)} = u_{2n-1}^{(2)} - u_{2n}^{(2)}$. For general $s \in \mathbb{N}$ we construct $A_s = \{u_n^{(s)}\}$ and $B_s = \{v_n^{(s)}\}$ in the same way and obtain

$$d_E(A_s) \leq \inf_n ||v_n^{(s)}|| + \varepsilon.$$

Now, the inclusion $A_{s+1} \subseteq B_s$ implies that

$$\inf_n ||v_n^{(s)}|| \leq \inf_n ||u_n^{(s+1)}|| \leq \overline{\lim_{n\to\infty}} ||u_n^{(s+1)}|| \leq r_E(A_{s+1});$$

therefore we have $d_E(A_s) \leq r_E(A_{s+1}) + \varepsilon$ and, since $r_E(K) \leq \frac{1}{2}J(E)d_E(K)$ for any bounded $K \subset E$,

$$1 \leq d_E(A_1) \leq \left(\frac{J(E)}{2}\right)^n d_E(A_{n+1}) + \varepsilon\sum_{k=0}^{n-1}\left(\frac{J(E)}{2}\right)^k.$$

In the last term we can make ε tend to zero, while in the other term the diameter $d_E(A_{n+1})$ can be expressed in the form

$$d_E(A_{n+1}) = \sup \Big\| \sum_{j\in I} x_j \Big\|,$$

where the supremum is taken over all index sets I with 2^n elements. From this we easily deduce (9).

To prove the last assertion, we can take $||x_j|| = 1$ in (9) and use the set $I = I^{(n)}$ with $|I| = 2^n$ also in (8). Then we choose a subsequence $\{n_j\}$ such that

$$\varlimsup_{n\to\infty} \Big\| \sum_{j\in I^{(n)}} x_j \Big\|^{1/n} = \lim_{i\to\infty} \Big\| \sum_{j\in I^{(n_i)}} x_j \Big\|^{1/n_i} = Q'(E).$$

Thus, $J(E) \geq \max\{Q'(E), 2/Q'(E)\} \geq \sqrt{2}$ as claimed. ∎

We point out that the estimate $J(E) \geq \sqrt{2}$ obtained above is not a consequence of Dvoretsky's theorem and the known equality

$$J(l_2) = \sup_n J(l_2^n) = \sqrt{2},$$

because in the definition of $r_E(A)$ the infimum is taken over E, not A. Indeed, it is not true that $J(F) \leq J(E)$ if F is a subspace of E. For example, $J(l_\infty) = 1$, but $J(c_0) = 2$. Finally, the example $J(L_\infty) = 1$ shows that the assumption that E does not contain c_0 cannot be dropped in Theorem 1.

COROLLARY 1: *If E satisfies a lower q-estimate $(1 \leq q < \infty)$ then*

$$J(E) \geq 2^{1/q}. \tag{11}$$

If E does not contain a subspace isomorphic to c_0 and satisfies an upper p-estimate $(1 \leq p < \infty)$ then

$$J(E) \geq 2^{1-1/p}. \tag{12}$$

Finally, suppose that there exists a sequence of disjointly supported normalized elements of E which is equivalent to the unit vectors of l_q. Then the lower estimate

$$J(E) \geq \max\{2^{1/q'}, 2^{1/q}\} \tag{13}$$

holds.

We point out that the estimates (11) and (12) are sharp because they turn into equalities for L_p-spaces. Note that the inequality (13) means, in particular, that $J(E) \geq J(l_q)$.

We also remark that the assumption on upper p-estimates and lower q-estimates in Corollary 1 (and in similar statements which follow) may be weakened. For instance, one may replace the constant C in (1) by $(\log n)^s$ for some $s > 0$, and the constant c in (2) by $(\log n)^{-t}$ for some $t > 0$.

Corollary 1 may be applied to compute $J_0(L_{p,q})$ as we shall show now.

THEOREM 2: *Let $1 < p < \infty$ and $1 \leq q < \infty$. Then the equality*

$$J_0(L_{p,q}) = \max\{2^{1/p}, 2^{1/p'}, 2^{1/q}, 2^{1/q'}\} \tag{14}$$

holds.

Proof: It is well known that $L_{p,q}$ satisfies a lower s-estimate with $s = \max\{p, q\}$. The sequence of functions $2^{k/p}\chi_{(2^{-k},2^{-k+1})}(t)$ $(k = 1, 2, \ldots)$ generates in $L_{p,q}$ a subspace which is isomorphic to l_q. For $q \leq p$, this statement is contained in [9]; however, it is true also for $p < q < \infty$, and the proof is similar. From Corollary 1 we conclude that

$$J_0(L_{p,q}) \geq \max\{2^{1/q}, 2^{1/q'}\}.$$

On the other hand, in [23] it has been proved that

$$\max\{2^{1/p}, 2^{1/p'}\} \leq J_0(L_{p,q}) \leq \max\{2^{1/p}, 2^{1/p'}, 2^{1/q}, 2^{1/q'}\}.$$

Combining these inequalities we arrive at (14). ∎

Suppose now that E is a p-convex and q-concave r.i. space. It is clear that this does not yet imply a nontrivial upper estimate for the Jung constant $J(E)$. However, the constant $J_0(E)$ may then be computed explicitly, as we shall show now. To this end, let

$$p(E) = \sup\{p : 1 \leq p \leq \infty,\ E \text{ is a } p\text{-convex lattice}\}$$

and

$$q(E) = \inf\{q : 1 \leq q \leq \infty,\ E \text{ is a } q\text{-concave lattice}\}.$$

By Theorem 1.f.7 of [15], $p(E)$ coincides with the supremum of all p such that E satisfies an upper p-estimate, and $q(E)$ coincides with the infimum of all q such that E satisfies a lower q-estimate.

THEOREM 3: *Suppose that E is a r.i. space which does not contain a subspace isomorphic to c_0. Then the equality*

$$J_0(E) = \max\{2^{1/p(E)}, 2^{1/q'(E)}\} \tag{15}$$

holds.

Proof: Since the estimate $J_0(E) \le 2$ is trivial, we may assume in the first part of the proof that E is p-convex for some $p > 1$ and q-concave for some $q < \infty$. Put $\theta = 2 \min\{1/p', 1/q\}$. By Pisier's theorem [20], there exists a Banach lattice F such that the spaces E and $[F, L_2]_\theta$ coincide up to equivalence. (Spaces E with this property are called **θ-Hilbert spaces**.) The construction of F given in [20] shows that F is a r.i. space, and hence $[F, L_2]_\theta$ is r.i., too. Applying the results of [23] we get the estimate

$$J([F, L_2]_\theta) \le 2^{1-\theta/2}.$$

Consequently, $J_0(E) \le \max\{2^{1/p}, 2^{1/q'}\}$, and thus

$$\begin{aligned} J_0(E) &\le \inf\{\max\{2^{1/p}, 2^{1/q'}\}\colon 1 \le p < p(E),\ q(E) < q < \infty\} \\ &= \max\{2^{1/p(E)}, 2^{1/q'(E)}\}. \end{aligned}$$

Let us now prove the inverse inequality. Given $r > p(E)$, we may find an increasing sequence of integers $\{n_j\}$ and a sequence of disjointly supported normalized elements $z_{j,i} \in E$ ($j = 1, 2, \dots$; $i = 1, 2, \dots, n_j$) such that

$$\|z_{j,1} + z_{j,2} + \dots + z_{j,n_j}\| \ge n_j^{1/r} \quad (j = 1, 2, \dots). \tag{16}$$

To see this, assume the contrary. Then the estimate

$$\Big\|\sum_{i \in I} x_i\Big\| \le |I|^{1/r} \tag{17}$$

is true for every finite index set $I \subset \mathbb{N}$ and every sequence of disjointly supported normalized elements $x_i \in E$. Consider the Lorentz sequence space $l_{r,1}$ endowed with the norm

$$\|a\|_{l_{r,1}} = \|(a_1, a_2, \dots)\|_{l_{r,1}} = \sum_{k=1}^{\infty} a_k^*(k^{1/r} - (k-1)^{1/r}),$$

where $\{a_k^*\}$ denotes the decreasing rearrangement of the sequence $\{|a_k|\}$. By Theorem 2.5.2 of [14], the estimate (17) implies that the operator T defined by

$$Ta = \sum_{k=1}^{\infty} a_k x_k$$

is bounded from $l_{r,1}$ into E with $\|T\| \le 1$. Now, from Hölder's inequality it follows that $l_{r,1} \supset l_s$ for $s < r$. Choosing, in particular, $p(E) < s < r$, we conclude that the operator T is also bounded from l_s into E. But this means that E in fact satisfies an upper s-estimate, contradicting the maximality of $p(E)$. This contradiction shows that (16) is true.

For every $j \in \mathbb{N}$ there exists an $m_j \in \mathbb{N}$ such that $2^{m_j} \leq n_j < 2^{m_j+1}$. Without loss of generality we may assume that

$$\Big\|\sum_{i=1}^{2^{m_j}} z_{j,i}\Big\| \geq \frac{1}{2} n_j^{1/r} \geq 2^{m_j/r-1}.$$

Using (10) and the estimate $J(E) \geq \sqrt{2}$ we get

$$J(E) \geq \lim_{r \to p(E)} \lim_{j \to \infty} \left(\frac{\sqrt{2}-1}{4} 2^{m_j/r} \right)^{1/m_j} = 2^{1/p(E)}. \tag{18}$$

In this way, we have established the lower estimate $J_0(E) \geq 2^{1/p(E)}$.

Let us prove the second part of the estimate. Given $r < q(E)$, we claim that we can find an increasing sequence of integers n_j and a sequence of disjointly supported normalized elements $y_{j,i} \in E$ $(j = 1, 2, \ldots; i = 1, 2, \ldots, n_j)$ such that

$$\Big\|\sum_{i=1}^{n_j} y_{j,i}\Big\| \leq n_j^{1/r} \quad (j = 1, 2, \ldots). \tag{19}$$

In fact, if this were false we would have

$$\Big\|\sum_{i \in I} x_i\Big\| \geq |I|^{1/r}$$

for every finite index set $I \subset \mathbb{N}$ and every sequence of disjointly supported normalized elements $x_i \in E$. By Theorem 2.5.7 of [14], this implies that, for every finite real sequence $a = (a_1, a_2, \ldots)$,

$$\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \geq ||a||_{l_{r,\infty}} = \sup_{I \subseteq \mathbb{N}} |I|^{1/r-1} \sum_{i \in I} |a_i|.$$

Since $l_{r,\infty} \subset l_s$ for $s > r$ we further obtain

$$\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \geq c(r,s)||a||_{l_s},$$

where $c(r,s)$ is the corresponding imbedding constant. Choosing, in particular, $r < s < q(E)$, we conclude that E in fact satisfies a lower s-estimate, contradicting the minimality of $q(E)$. This contradiction shows that (19) is true.

Now, taking m_j with $2^{m_j} \le n_j < 2^{m_j+1}$ we have

$$\left\| \sum_{i=1}^{2^{m_j}} y_{j,i} \right\| \le 2^{(m_j+1)/r}.$$

Combining this with (9) we get

$$J(E) \ge 2 \lim_{j\to\infty} \left\| \sum_{i=1}^{2^{m_j}} y_{j,i} \right\|^{-1/m_j} \ge 2 \lim_{j\to\infty} \left(2^{-(m_j+1)/r}\right)^{1/m_j} = 2^{1-1/r},$$

hence $J_0(E) \ge 2^{1/q'(E)}$, which proves (15). ∎

It is well known (see e.g. Proposition 1.d.4 of [15]) that the notions of convexity and concavity are dual to each other. This leads to the following

COROLLARY 2: *Let E be a reflexive r.i. space. Then the equality*

$$J_0(E) = J_0(E^*) \tag{20}$$

holds.

We point out that, in general, the numbers $J_0(E)$ and $J_0(E^*)$ may be different; for example, $J_0(L_1) = 2$, but $J_0(L_\infty) = 1$. However, according to the best of our knowledge this is the only example where the equality (20) fails.

Let E be a r.i. space and $r \ge 1$. The space $E(r)$ endowed with the norm

$$||x||_{E(r)} = ||\,|x|^r\,||_E^{1/r}$$

is called the r-**convexification** of E (see e.g. §1.d of [15]). It is easy to check that $p(E(r)) = rp(E)$ and $q(E(r)) = rq(E)$. This simple observation allows us to derive from Theorem 3 the following

COROLLARY 3: *Suppose that E does not contain a subspace isomorphic to c_0, and let $r > 1$. Then the equality*

$$J_0(E(r)) = \max\{2^{1/rp(E)}, 2^{1-1/rq(E)}\}$$

holds. In particular, $J_0(E(r)) < 2$ if E is q-concave for some $q < \infty$.

As mentioned above, the lower estimate $J_0(E) \ge \sqrt{2}$ holds for every r.i. space $E \ne L_\infty$. Moreover, in [23] it was proved that, among all r.i. spaces E satisfying a lower q-estimate for some $q < \infty$, the space L_2 is the only space for which $J(E) = \sqrt{2}$. However, this is not true if we replace the constant $J(E)$ by the constant $J_0(E)$.

For example, let $E = L_{M_\lambda}$ be the Orlicz space generated by the Young function $M_\lambda(u) = u^2[\log(1+|u|)]^\lambda$ for some $\lambda \geq 0$. Of course, $L_{M_0} = L_2$, and $L_{M_\lambda} \neq L_{M_\mu}$ for $\lambda \neq \mu$. A straightforward calculation shows that $p(L_{M_\lambda}) = q(L_{M_\lambda}) = 2$ for every $\lambda \geq 0$, hence $J_0(L_{M_\lambda}) \equiv \sqrt{2}$.

To illustrate the above results, we give some further examples. First, we point out that it is impossible to obtain a nontrivial upper estimate for the constants $J(E)$ and $J_0(E)$ in the class of reflexive r.i. spaces. To see this, consider the space F of all measurable functions for which the norm

$$||x||_F = \left(\int_0^1 x^*(t)^2 \frac{dt}{t(\log \frac{1}{t} + 1)^2}\right)^{1/2}$$

is finite. By construction, F is the 2-convexification of the Lorentz space E defined by the norm

$$||x||_E = \int_0^1 x^*(t) \frac{dt}{t(\log \frac{1}{t} + 1)^2}.$$

By [16], F is reflexive. Since

$$\lim_{\tau \to 0} \frac{||\chi_{(0,2\tau)}||_F}{||\chi_{(0,\tau)}||_F} = \lim_{\tau \to 0} \frac{\sqrt{\log(\frac{1}{\tau} + 1)}}{\sqrt{\log(\frac{1}{2\tau} + 1)}} = 1,$$

from what has been proved in [23] we conclude that $J_0(F) = J(F) = 2$.

However, if we study the problem of finding nontrivial upper estimates within the class of Orlicz spaces, Theorem 3 gives the following

COROLLARY 4: *If L_M is a reflexive Orlicz space, then $J_0(L_M) < 2$.*

In fact, Proposition 2.b.5 of [15] implies that $p(L_M) > 1$ and $q(L_M) < \infty$. Applying Theorem 3 above the assertion follows.

Recall that the reflexivity of an Orlicz space L_M is equivalent to the fact that both the Young function M and its conjugate Young function M^* (see e.g. [13, 21]) satisfy a Δ_2-estimate.

Observe that the constant $J_0(L_M)$ in Corollary 4 must not be replaced by the Jung constant $J(L_M)$. For example, choose $M(u) = \max\{2|u|, u^2\}$ and $A = \{r_1, r_2, r_3, \ldots\}$, where $r_n(t) = \text{sign} \sin 2^n \pi t$ are the Rademacher functions. Then

$$r_{L_M}(A) = d_{L_M}(A) = ||r_1||_{L_M} = 2,$$

hence $J(L_M) = 2$. On the other hand, $J_0(M) = \sqrt{2}$, because $L_M = L_2$ up to equivalence. We mention the article [5], where a similar problem was studied.

Here is yet another example. Let $1 \le p \le q \le \infty$, and consider the space $E = L_p \cap L_q$ over $(0, \infty)$ equipped with the norm $\|x\|_{L_p \cap L_q} = \max\{\|x\|_{L_p}, \|x\|_{L_q}\}$. It is clear that $L_p \cap L_q$ is a p-convex and q-concave Banach lattice. Since the finiteness of the underlying measure is not essential in Theorem 3, we arrive at the following

COROLLARY 5: *Let $1 \le p \le q < \infty$. Then the equality*

$$J_0(L_p \cap L_q) = \max\{2^{1/p}, 2^{1/q'}\} \tag{21}$$

holds.

Let us return to Theorem 2 which provides an explicit formula for $J_0(L_{p,q})$ in case $1 < p < \infty$ and $1 \le q < \infty$. Unfortunately, neither Corollary 1 nor Theorem 3 applies to the case $q = \infty$, since the space $L_{p,\infty}$ contains a subspace isomorphic to c_0. In the next theorem, however, we give a partial result in this direction.

THEOREM 4: *Let $1 < p < \infty$, and let $A = \{x_1, x_2, \ldots\}$, where x_i is a disjointly supported sequence of functions in $L_{p,\infty}$. Then there exists a constant $c \in (\frac{1}{2}, 1)$ depending only on p such that*

$$r_{L_{p,\infty}}(A) \le c\, d_{L_{p,\infty}}(A).$$

Proof: Without loss of generality we may assume that the supports of x_i are disjoint intervals (a_i, b_i), x_i is a decreasing positive function on (a_i, b_i), and

$$d_{L_{p,\infty}}(A) \le 1. \tag{22}$$

For $i = 1, 2, \ldots$, we put

$$u_i(t) = \begin{cases} \min\{x_i(t),\ (2(t - a_i))^{-1/p}\} & \text{if} \quad t \in (a_i, b_i), \\ 0 & \text{if} \quad t \notin (a_i, b_i), \end{cases}$$

and $v_i = x_i - u_i$. Given $\tau > 0$, the set of all $i \in \mathbb{N}$ for which

$$\operatorname{mes}\{t :\ x_i(t) \ge \tau\} > \tfrac{1}{2}\tau^{-p} \tag{23}$$

contains at most one element. Indeed, if (23) is valid for $k, l \in \mathbb{N},\ k \ne l$, then

$$\|x_k - x_l\|_{L_{p,\infty}} \ge \tau\big(\operatorname{mes}\{t :\ x_k(t) \ge \tau\} + \operatorname{mes}\{t :\ x_l(t) \ge \tau\}\big)^{-p} > 1,$$

contradicting our assumption (22). We conclude that, for fixed $t > 0$, the set $\{i :\ x_i^*(t) > (2t)^{-1/p}\}$ contains at most one element. Therefore, the two functions

$$v(t) = \sum_{i=1}^{\infty} v_i(t), \quad w(t) = \sup_{i \in \mathbb{N}} \max\{x_i^*(t) - (2t)^{-1/p}, 0\}$$

are equimeasurable. Since $x_i^*(t) \le t^{-1/p}$ for every $t > 0$ and $i = 1, 2, \dots$, we have $v^*(t) \le (1 - 2^{-1/p})t^{-1/p}$. For every $j \in \mathbb{N}$, the functions $\hat{v}_j(t) = v(t) - v_j(t)$ and $u_j(t)$ are disjointly supported. Consequently,

$$\operatorname{mes}\{t : \ \hat{v}_j(t) + u_j(t) > \tau\} \le \big((1 - 2^{-1/p})^p + \tfrac{1}{2}\big)\tau^{-p}$$

for each $\tau > 0$. This means that

$$\|\hat{v}_j + u_j\|_{L_{p,\infty}} \le \big((1 - 2^{-1/p})^p + \tfrac{1}{2}\big)^{1/p},$$

hence $\|x_j - v\|_{L_{p,\infty}} = \|\hat{v}_j + u_j\|_{L_{p,\infty}} \le c$ for every $j \in \mathbb{N}$, where

$$c = c(p) = \big((1 - 2^{-1/p})^p + \tfrac{1}{2}\big)^{1/p}.$$

It is evident that $\frac{1}{2} < c(p) < 1$ for every $p \in (1, \infty)$, and thus everything is proved. ∎

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